IOP Publishing European Journal of Physics
Eur. J. Phys. 39 (2018) 025806 (8pp) https://doi.org/10.1088/1361-6404/aa9e7b

Unified presentation of four fundamental inequalities

Joseph Lajzerowicz[1], **Roland Lehoucq**[2] and **François Graner**[3]

[1] Retired from Laboratoire Interdisciplinaire de Physique, CNRS UMR 5588, Université Grenoble Alpes, Saint-Martin d'Hères, France
[2] Laboratoire AIM-Paris-Saclay, CEA/DSM/Irfu, CNRS UMR 7158, Université Paris Diderot, CEA Saclay, Gif-sur-Yvette, France
[3] Laboratoire Matière et Systèmes Complexes, CNRS UMR 7057, Université Paris Diderot, Paris, France

E-mail: francois.graner@univ-paris-diderot.fr

Received 17 June 2017, revised 3 November 2017
Accepted for publication 30 November 2017
Published 15 February 2018

Abstract

We suggest an unified presentation to teach fundamental constants to graduate students, by introducing four lower limits to observed phenomena. The reduced Planck constant $\hbar$ is the lowest classically definable action. The inverse of invariant speed, s, is the lowest observable slowness. The Planck time, t_P, is the lowest observable time scale. The Boltzmann constant, k, determines the lowest coherent degree of freedom; we recall an Einstein criterion on the fluctuations of small thermal systems and show that it has far-reaching implications, such as demonstrating the relations between critical exponents. Each of these four fundamental limits enters in an inequality, which marks a horizon of the Universe we can perceive. This compact presentation can resolve some difficulties encountered when trying to defining the epistemologic status of these constants, and emphasizes their useful role in shaping our intuitive vision of the Universe.

Keywords: fundamental constants, Planck constant, Planck time, Boltzmann constant, light slowness, Einstein criterion, critical exponents

1. Fundamental constants

The so-called 'natural' [1, 2], 'universal' [3, 4] or 'fundamental' [5, 6–8] constants have a controversial status, and even their number is controversial. We will not attempt at reviewing the huge literature on the subject (see e.g. [6] and references therein). It suffices here to mention that in books, articles, seminars or classes, one encounters for instance the four following constants.

The reduced Planck constant, $\hbar$, relates the frequency and energy exchanged by an oscillator. The invariant speed, c, or speed of light in vacuum, relates invariant mass and rest energy. The constant of gravitation, G, relates the masses of interacting objects with their Newtonian interaction force, or the relativistic energy-momentum tensor with the space-time curvature. The Boltzmann constant, k, relates the temperature of a system and the energy of its thermal fluctuations, or the thermal fluctuations of a system and its dissipation, or the entropy of a system and the number of its accessible microscopic states.

These constants have a dimension. They can be all taken as equal to one by redefining the system of units. Thus the importance of these constants does not lie in their actual value, but rather in the fact that they enter in important equations which relate very different domains [9]. We do not discuss here their possible variation, especially given the fact that c is fixed and, with the second, defines the meter in the SI units system. In 2018 $\hbar$ and k become fixed too [5, 10, 11] in order to define the kilogram and the kelvin. More generally, the number of the constants, their combinations and the prefactors are conventional choices.

Let us immediately make clear that 'coupling constants' are different. The coupling constants are dimensionless, and far more numerous. They include for instance Sommerfeld's fine-structure constant, the strong coupling constant, and other coupling constants which describe interactions. The fact that their value is smaller or larger than 1 has an intrinsic meaning and defines different regimes. Thus coupling constants and fundamental constants should not be confused.

2. Fundamental limits and inequalities

2.1. A set of four limits

We suggest that the interest of the fundamental constants lies in their role in an *inequality*, which evidences essential limits of the Universe we can perceive. Some of these limits have been made intuitive by Gamow's tales [12], where a character called 'Mr Tompkins' undergoes surprising adventures because the values of the fundamental constants change. In fact, these constants delimitate the realm of scientific studies, whether current or future, from that of metaphysical speculations. Far from these limits is the realm of classical physics. Thus, it might be appropriate to define them as 'fundamental limits' entering 'fundamental inequalities'. This definition might turn more robust to technical progresses than definitions [3] based on horizons related with our current capacity to perceive the Universe.

Since the choice of fundamental limits is arbitrary, we are entitled to motivate our choice by pedagogy and elegance, or even by reference to poetry (*les quatre horizons qui crucifient le monde*, the four horizons that crucify the world [13]). We select four limits which each enter in a fundamental inequality (see table 1):

- The Planck constant over 2π, $\hbar$, is the lowest value of classically definable action, that is, angular momentum, or product of energy with time.
- The invariant speed, c, is the maximal physical speed, reached by massless objects and only by them, and the maximal speed for causal signals. In order to introduce a lower rather than an upper limit, we will use its inverse, $s = c^{-1}$, which can be called the 'invariant slowness', where the useful notion of slowness is defined and discussed below.
- The Planck frontier is the lowest observable scale, where 'scale' can refer to time, length or energy scale, as discussed below.
- The Boltzmann constant k enters in the definition of the lowest coherent degree of freedom.

Table 1. Four fundamental limits and associated constants. These values and their uncertainty (in brackets) have been listed in 2014 [4]. The value of s is fixed and thus without uncertainty. With the new base for SI units in 2018, values for $\hbar$ and k become fixed too [5, 10, 11].

Constant	*Value and unit*	*Limit*	*Limit name*	*Theoretical domain*
$\hbar$	$1.054\ 571\ 800(13) \times 10^{-34}$ J s	Action	Heisenberg principle	Quantum physics
s	$3.335\ 640\ 952 \ldots \times 10^{-9}$ s m^{-1}	Slowness	Einstein causality	Special relativity
t_P	$5.391\ 16(13) \times 10^{-44}$ s	Scale	Planck frontier	Scale relativity
k	$1.380\ 648\ 52(79) \times 10^{-23}$ J K^{-1}	Coherence	Einstein fluctuations	Statistical physics

The Planck constant is probably the most consensual one (the division of h by 2π, introduced by Dirac, is more than a handy convention [14]). We now comment briefly the notion of slowness (section 2.2) and the choice of the Planck frontier (section 2.3), then in more details the physical role of Boltzmann constant and its applications (section 3).

2.2. Slowness

Slowness is the inverse of speed. This interesting notion would deserve a separate article. Teaching essential limits and teaching slowness mutually reinforce each other and could be done in any order. For instance, one could use table 1 as a mean to introduce the notion and usefulness of slowness. Or, after having explained slowness in daily life, it could be used to introduce naturally table 1.

In daily life, few persons care for the addition of speeds; when you walk inside a plane, boat or train, or on a treadmill, you seldom need to estimate your velocity with respect to the ground (you can leave that as an exercise for physics students). On the opposite, what we routinely use in daily life is the addition of durations required to go from a given place to another one, and thus of the distances multiplied by slowness. Sportspersons like runners claim their times rather than their speeds; they record 'minutes per kilometer': this quantity is additive; its time integral has a physical meaning, which is the total duration. Sismologists use slowness because they compose different materials with different densities and hence different acoustic indices [15].

Slownesses compose according to the relativistic law of composition, just as the speeds do. Slowness might be more fundamental than speed: it is analogous with optical indices and is part of Fermat, Huygens and Maupertuis least action principles [16]. For instance for ray refraction at an interface, the Huygens construction for rays and wavefronts duality uses slowness and transposes in the Fourier space ('dispersion surface', i.e. surface of wavevectors) what the Descartes construction performs with speed in real space.

Following Malus, given an initial position $\vec{r}_0$, one can define the 'duration' function $d(\vec{r}) = T(\vec{r}_0, \vec{r})$ which is the time T required for an object starting from $\vec{r}_0$ to reach the position $\vec{r}$. The gradient of the function d is a vector, with the unit of an inverse speed, which defines the vectorial slowness: $\vec{\mathcal{L}} = \nabla d$. For instance, when a point source emits light, the wave surfaces are sets of points with same d; then $\vec{\mathcal{L}}$ is perpendicular to these wave surfaces and parallel to the wave vector. The scalar product of slowness and speed vectors is $\vec{\mathcal{L}}.\vec{v} = 1$ [16]. Optimising a boat trajectory requires to direct the slowness vector $\vec{\mathcal{L}}$ (rather than the speed $\vec{v}$) towards the goal [16].

Here $s = c^{-1}$ is the slowness of light in vacuum, the minimal physical slowness, reached by massless objects and only by them, and the minimal slowness for causal signals. Without any reference to these classical significations, s can also be defined as the invariant slowness in changes of inertial frames [17, 18]. Note that s^2 is the single parameter left free by the symmetries in the derivation of the Lorentz transformation by Lévy-Leblond [17].

2.3. Planck frontier

The Planck frontier can be associated with a time scale $t_P = (\hbar G/c^5)^{1/2}$, a length scale $\ell_P = (\hbar G/c^3)^{1/2}$, or an energy scale $E_P = (\hbar c^5/G)^{1/2}$. These Planck time, length and energy arise immediately from dimensional analysis from the Planck constant, even within Newtonian gravitation. However, their modern interpretation as a frontier resulting from an inequality is due to Bronstein, in 1936, and is rooted in general relativity [19–22], as follows.

The Planck frontier characterises the competition between gravitational binding and limitations to confinment in quantum mechanics. Beyond the Planck frontier, quantum fluctuations of gravitational field are so large that space-time is no longer a continuous differential variety. As Rovelli and Vidotto write it [22], Bronstein's above original argument is that space and time are necessarily ill-defined in quantum gravity. The solution is to accept that observables do not resolve space and time more finely than Planck scale. This forces the connection between gravity and geometry and is the core of modern attempts towards quantum gravity.

Equivalently, an object which Compton length is equal to ℓ_P is a mini blackhole. Quantitatively, this amounts to writing that for an object of mass m, the Schwarzschild radius is $r = 2Gm/c^2$, and the object is at most confined over a size equal to its Compton length, $r = \hbar/2mc$. Eliminating m yields $r^2 = \hbar G/c^3$, and hence the Planck size, $r = \ell_P$; the Planck time t_P is derived immediately. Alternatively, eliminating r yields the Planck mass; the Planck energy E_P is derived immediately. This derivation uses concepts that predate Bronstein (Schwarzschild radius in 1916, Compton effect in 1923). It can even be formulated in a Newtonian framework, where the escape speed is c if the object has a size $r = 2Gm/c^2$: surprisingly, even the prefactor is correct.

There could be arguments to favor the Planck length ℓ_P; for instance, scale relativity theories invoke an invariant scale, which is more often presented as a length than as a duration [22, 23]. But we suggest to use the Planck time, t_P, which is the lowest observable time scale. In fact, we find this choice is pedagogical for graduate students, because the time is a true scalar while space is more associated with vectors; and because Planck energy is an upper limit rather than a lower one. Also, note that in popular science the Big Bang scenario is usually more described in terms of time than of length or of energy. Finally, since 1983 the durations are conventionnally defined as more fundamental quantities than the lengths (which are now derived from durations through c).

3. Boltzmann constant

It has been argued that k is not fundamental in the sense that its value can change without fundamental affecting physical phenomena [7]. In 2014, the NIST website lists the Boltzmann constant simply as a 'frequently used' constant. What it lists as a 'fundamental' constant is rather the Planck temperature, T_P, defined as the ratio of the Planck energy to the Boltzmann constant, E_P/k [4].

On the opposite, there exist arguments in favor of its fundamental status. If a modern Gamow wrote the story of a country where k is high, animals would become Brownian, especially smaller ones, and hunting might turn as difficult as in a quantum country [12]. With the new base for SI units in 2018, k becomes fixed [5, 10, 11]. It enters into a fundamental inequality, as we will now explain in detail.

3.1. Gibbs and Einstein estimation of fluctuations

The energy E of a system at temperature T fluctuates. Gibbs [24] showed that the variance $\sigma^2 = \langle E^2 \rangle - \langle E \rangle^2$ of this energy obeys:

$$\sigma^2 = kT^2 \frac{\partial E}{\partial T}, \tag{1}$$

where $\langle E \rangle$ is the average over fluctuations, and $\partial E / \partial T$ is the system's heat capacity corresponding to the existing constraints (such as constant pressure or constant volume). We check that both sides of equation (1) are insensitive to an additive constant in the energy.

Equation (1) is largely independent from the hypotheses used to derive it. For instance, Einstein later re-demonstrated it independently [25, 26]. His demonstration goes as follows (he denoted by 2κ what we note k, and we refer to a version where typographic mistakes have been corrected [27]). Since $\langle E \rangle = \int E\ \mathcal{P}(E) \mathrm{d}E$ where the probability of the value E is $\mathcal{P}(E) \propto \exp(-E/kT)$, one can simply write

$$\int (E - \langle E \rangle) \exp^{-\frac{E}{kT}} \mathrm{d}E = 0. \tag{2}$$

Differentiating the left hand side of equation (2) with respect to T yields equation (1).

3.2. A fundamental inequality: Einstein criterion

Gibbs remarks that the energy is extensive, so that E is proportional to the volume V of the system under consideration; thus σ goes as $V^{1/2}$ and is much smaller that E in the limit of large volumes. Einstein similarly remarks that, on the opposite limit, if the system size V is small enough, σ becomes comparable to E, or even larger than E [25]. Without expliciting his thought, he concludes that k plays a role in the system stability. Einstein criterion states there is a critical volume such that

$$E > \sigma. \tag{3}$$

It can be written, using equation (1), as a fundamental inequality involving k:

$$\left(\frac{E}{T}\right)^2 \left(\frac{\partial E}{\partial T}\right)^{-1} > k. \tag{4}$$

Using the energy density $e = E/V$, equation (4) can be equivalently rewritten as

$$V > V_s, \tag{5}$$

where

$$V_s = k \frac{\partial e}{\partial T} \left(\frac{e}{T}\right)^{-2} \tag{6}$$

and $\partial e / \partial T$ is the volume-specific heat capacity. The physical interpretation of V_s is the volume which energy is equal to its thermal fluctuations (which might impact on its stability); or a coherent volume, which defines the effective thermodynamical degree of freedom.

3.3. Einstein's applications of his criterion

Einstein remarks, without providing explanations, that this criterion is difficult to apply in practice [25]; and he suggests one application, namely to the blackbody. He finds that $V_s^{1/3} T = 0.42\ 10^{-3}$ m. Thus $V_s^{1/3}$ correctly yields the wavelength of the blackbody radiation maximum (Wien's law), inversely proportional to the temperature, $\lambda_m = \sigma_\mathrm{W}/T$, with even the good order of magnitude for the Wien constant, $\sigma_\mathrm{W} = 0.2898\ 10^{-3}$ m K^{-1} (in his article, Einstein used the value 0.293). This mere 43% overestimating is a good agreement and, given the generality of the hypotheses involved, it is probably not a coincidence, says Einstein.

Alternatively, we note that $V_s^{1/3}$ and λ_m have the same order of magnitude as the correlation length of fluctuations in a gas of photons at thermal equilibrium, $\hbar c/kT$ [28].

As mentioned above, both sides of equation (1) are insensitive to an additive constant in the energy. However, equation (3) does depend on an additive constant in the energy, and implicitly assumes that there is a reference state with zero energy. We suggest that one important difficulty in applying this criterion resides in the necessity to define without ambiguity such a reference energy. That might explain why Einstein applies it to the blackbody radiation: in this case, the reference energy is intrinsically zero, representing the space devoid of photons.

3.4. Other applications of Einstein's criterion

We argue that other applications of equation (1) include the determination of scaling laws in phase transitions. Both sides of equation (3) diverge when approaching the transition, but they diverge together, and it is this concomitant divergence that we examine. This has been briefly done by one of us (p 1211, 1214, 1215 of [29]). We can rephrase the argument in a more general form, as follows. The critical exponent for energy density is α, defined by

$$\begin{aligned} e &\propto t^{1-\alpha} \\ \frac{\partial e}{\partial T} &\propto t^{-\alpha}, \end{aligned} \tag{7}$$

where $t = (T - T_c)/T_c$ is the reduced temperature near the transition temperature T_c. The correlation length ℓ_c scales with an exponent ν:

$$\ell_c \propto t^{-\nu}. \tag{8}$$

So the correlation volume goes as ℓ_c^D, where D is the dimension of space:

$$V \propto t^{-D\nu} \tag{9}$$

Equations (1) and (6) yield $-D\nu - \alpha = -2D\nu + 2(1 - \alpha)$, i.e.

$$\alpha + D\nu = 2. \tag{10}$$

This is one of the scaling relations between critical exponents [28]. The other scaling relations can all be derived similarly, in this simple, general, unified way which is pedagogical for graduate students.

Replacing the energy with the magnetization, and the specific heat with the magnetic susceptibility, equation (1) yields the Ginzburg-Levanyuk inequality which marks the onset of the critical regime [28, 29]. Similarly, the Debye length in ionic solutions can be interpreted as a length marking the competition between potential energy (which groups the charges together) and fluctuations (entropic dispersion of the charges); in this example, where the energy vanishes when T goes to infinity, applying Einstein criterion is subtle.

4. Conclusion

The name 'Bronstein cube' is sometimes given to the 'cube of physical theories', graphically representing c, G and $\hbar$ as three axes [7, 20, 30]. This 'cube' becomes an hypercube if one also includes k (see [2] and references therein). We suggest that ($\hbar$, s, t_{P}, k) could be the axes of a hypercube, which would encompass the current set of physical theories and their fundamental limits.

Acknowledgments

We thank Marcel Vallade for the idea that fundamental constants correspond to inequalities, Jean-Philippe Uzan for stimulating discussions, Boris Bolliet for references, Jean-Marc Lévy-Leblond for critical reading of the manuscript.

ORCID iDs

François Graner https://orcid.org/0000-0002-4766-3579

References

[1] Barrow J D 2003 *The Constants of Nature: From Alpha to Omega* (New York: Vintage)
[2] Okun L 2002 Cube or hypercube of natural units *Multiple Facets of Quantization and Supersymmetry* (*Michael Marinov Memorial*) ed M Olshanetsky and A Vainshtein (Singapore: World Scientific)
[3] Cohen-Tannoudji G 1991 *Les Constantes Universelles* (Paris: Hachette) (in French)
[4] National Institute for Standards and Technology (NIST) 2014 CODATA internationally recommended values http://physics.nist.gov/cuu/Constants/
[5] Bureau International des Poids et Mesures (BIPM) http://bipm.org/en/
[6] Uzan J-P 2003 The fundamental constants and their variation: observational and theoretical status *Rev. Mod. Phys.* **75** 403–55
[7] Uzan J-P and Lehoucq R 2005 *Les Constantes Fondamentales* (Paris: Belin) (in French)
[8] Uzan J-P and Leclercq B 2008 *The Natural Laws of the Universe* (Heidelberg: Springer)
[9] Lévy-Leblond J-M 1977 On the conceptual nature of the physical constants *Riv. Nuovo Cimento* **7** 187–214
[10] Knotts S, Mohr P J and Phillips W D 2016 An introduction to the new SI *Phys. Today* **55** 16–21
[11] Newell D 2016 A more fundamental International System of Units *Phys. Today* **67** 35–41
[12] Gamow G 1946 *Mr Tompkins in Wonderland* (London: Macmillan)
[13] Jammes F 1913 Rosaire *L'Eglise habillée de feuilles* (Paris: Mercure de France) in French
[14] Lévy-Leblond J-M 2002 The meanings of Plancks constant *One Hundred Years of h* (*Conf. Proc. Soc. Ital. di Fisica* 79) ed E Beltrametti *et al* (Bologna: Società Italiana di Fisica) pp 211–26
[15] Kennett B L N 1981 Slowness techniques in seismic interpretation *J. Geophys. Res.* **86** 11575–84
[16] Kimball J C and Story H 1998 Fermat's principle, Huygens' principle, Hamilton's optics and sailing strategy *Eur. J. Phys.* **19** 15–24
[17] Lévy-Leblond J-M 1976 One more derivation of the Lorentz transformation *Am. J. Phys.* **44** 271–7
[18] Pelissetto A and Testa M 2015 Getting the Lorentz transformations without requiring an invariant speed *Am. J. Phys.* **83** 338–40
[19] Bronstein M P 1936 Kvantovanie gravitatsionnykh voln (Quantization of gravitational waves) *Zh. Eksp. Tear. Fiz.* **6** 195–236 (in Russian)
[20] Bronstein M P 1936 Quantentheorie schwacher Gravitationsfelder *Phys. Z. Sowjetunion* **9** 140–57 (in German)
[21] Gorelik G and Frenkel V 1994 *Matvei Petrovich Bronstein and Soviet Theoretical Physics in the Thirties* (Birkhauser Verlag)
[22] Rovelli C and Vidotto F 2015 *Covariant Loop Quantum Gravity—An Elementary Introduction to Quantum Gravity and Spinfoam Theory* (Cambridge: Cambridge University Press)
[23] Nottale L 1993 *Fractal Space-Time and Microphysics* (Singapore: World Scientific)
[24] Gibbs J W 1902 *Elementary Principles in Statistical Mechanics* (New York: Charles Scribner's Sons)
[25] Einstein A 1904 Zur allgemeinen molekularen Theorie der Wärme *Ann. Phys.* **34** 354–62 (in German)

Reprinted in Einstein A 2005 Commemorating Albert Einstein *Ann. Phys.* Special Issue **14** 154–63 and also in Einstein A 2005 *Einstein's Annalen Papers* ed J Renn (New York: Wiley-VCH)

[26] Peliti L and Rechtman R 2017 Einstein's approach to statistical mechanics: the 1902–04 papers *J. Stat. Phys.* **167** 1020–38

[27] Einstein A 1989 Œuvres choisies *Collection Sources du savoir* ed F Balibar (Paris: Seuil/CNRS) (in French)

[28] Landau L D and Lifshitz E M 2013 Statistical physics *Course of Theoretical Physics* (Amsterdam: Elsevier)

[29] Bastie P, Lajzerowicz J and Schneider J R 1978 Investigation of the ferroelectric-ferroelastic phase transition in KH_2PO_4 and RbH_2PO_4 by means of γ-ray diffractometry *J. Phys. C: Solid State Phys.* **11** 1203–16

[30] Gamov G, Ivanenko D and Landau L 1928 Mirovye postoyannye i predel'nyi perekhod *Zh. Russkogo Fiz.-Khim. Obshchestva* **60** 13–7 (in Russian)