Institute of Physics Publishing European Journal of Physics
Eur. J. Phys. **24** (2003) 267–270 PII: S0143-0807(03)58238-1

About the magnetic field of a finite wire

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Received 14 January 2003
Published 7 March 2003
Online at stacks.iop.org/EJP/24/267

Abstract
The Ampère theorem and the Biot–Savart law are well known tools used to calculate magnetic fields created by currents. Their use is not limited to the case of magnetostatics; they can also be used in time-dependent problems. We show in this paper that the highly classical example of a straight wire, generally treated as a simple magnetostatics problem, should be considered in the framework of time-varying fields. This academic question is a nice illustration showing the generality of the Biot–Savart law, and especially how it implicitly takes into account the charge conservation law.

1. Introduction

The Ampère theorem and the Biot–Savart law are well known tools used to calculate magnetic fields created by current distributions [1]. The former is often used in high-symmetry problems of magnetostatics, but it may be used in cases of time-varying field, with some precautions. The latter is more general, because it has fewer symmetry constraints. It can be used in a wide range of classical magnetism problems, and it provides an elegant way to calculate the magnetic fields created by plane current-carrying wires [2]; it can also be generalized to the electrostatic field [3]. Analogous equations can also be found in various physical problems, such as those of topological defects at the Kosterlitz–Thouless transition [4], magnetic fluids [5], amphiphilic monolayers [6], type-I superconductors [7], and the N-body problem of celestial mechanics [8].

Students usually hesitate over which one they should use. In particular, in the unphysical but simple case of a constant electric current I flowing through a *finite* straight wire (figure 1), students wonder why the two methods yield different results (see below—equations (1), (3)). Our experience shows that they find satisfactory the explanations that we develop in this paper. It is a clear presentation of how the charge conservation is implicitly taken into account by the Biot–Savart law.

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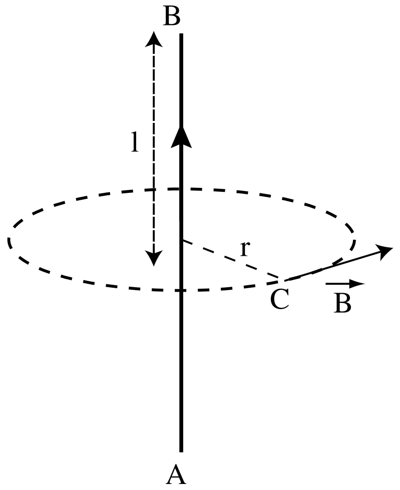

Figure 1. An electrical current I flows across the rectilinear *finite* wire AB. What magnetic field does it create at point C?

2. Magnetic fields of infinite and finite wires

The determination of the magnetic field created by a rectilinear *infinite* wire is a classical problem that can be found in every electromagnetism book. The calculation is generally carried out by applying the Ampère theorem to a current circulation on a circle Γ of radius r. Simple symmetry considerations indicate that the magnetic field is orthoradial with $\vec{B} = B(r)\vec{u}_\theta$, and the use of the Ampère law leads to the well known result

$$\mu_0 I = \oint_\Gamma \vec{B} \cdot \mathrm{d}\vec{l} = 2\pi R|\vec{B}(r)| \Rightarrow \vec{B}(C) = \frac{\mu_0 I}{2\pi R}\vec{u}_\theta. \tag{1}$$

The case of a *finite* wire AB of length $2l$ (figure 1) is also classical and generally found as an example of the use of the Biot–Savart law:

$$\vec{B}(\vec{r}) = \oint_{A\to B} \frac{\mu_0 I}{4\pi} \frac{\vec{r} \times \mathrm{d}\vec{s}}{r^3}. \tag{2}$$

This expression allows one to calculate the field at any point, and especially in the symmetry plane of the wire, leading to

$$\vec{B}(C) = \frac{\mu_0 I \sin\alpha}{2\pi r}\vec{u}_\theta = \frac{\mu_0 I}{2\pi} \frac{1}{\sqrt{r^2 + l^2}}\vec{u}_\theta. \tag{3}$$

Expression [3] appears to have the correct behaviour: it depends on the length of the wire, and gives the magnetic field of an infinite wire (equation (1)) for $r \ll 2l$. It is usually used to calculate the magnetic field in more complex situations, for example that of a squared closed loop.

3. The problem

Now we could also try to use the Ampère theorem to calculate the field in the symmetry plane of the *finite* wire. This leads to the same expression as for an infinite wire (equation (1)): surprisingly, it is independent of the wire's length; hence it is clearly different from the expression obtained by using the Biot–Savart law (equation (3)).

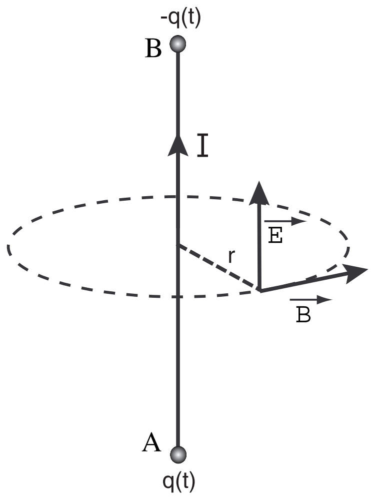

Figure 2. An electrical current I flows across the rectilinear *finite* wire AB. A source and a sink of electrical charge $+q$ and $-q$ ensure charge conservation.

Obviously, the Biot–Savart law gives the correct answer to our problem, and the Ampère law gives a wrong result. The question that we now have to answer is the following: what kind of mistake are we making by using Ampère's law in this case?

4. Solution

One way to answer this question is to say that a current I cannot flow along a finite wire: it is not a physical situation, because the wire has to be connected to something. This is not a problem in the case of an infinite wire, because one can suppose that the wire is closed at infinity.

This answer is not wrong, but is not totally satisfying, and students often demand more. The Biot–Savart law (equation (2)) is nothing more than the addition of contributions coming from many small wire elements. Why is it possible to calculate the magnetic field in an unphysical situation with the Biot–Savart law, and not with the Ampère theorem?

A solution giving physical sense to the finite wire is to put a source and a sink of electrical charge $+q$ and $-q$ at each extremity. It is then possible to have an electrical current I flowing along the wire (figure 2). The major point is now that both charges $q(t)$ and $-q(t)$ are time dependent and create an electrical field $\vec{E}(\vec{r})$ which is time dependent! Hence this problem is no longer a magnetostatics problem: we have to treat it in the more general framework of electromagnetism.

It is still possible to use Ampère's theorem, as long as we do that in a more general way. Using the Maxwell–Ampère law for time-varying fields leads to

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{j} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}. \tag{4}$$

By taking the flux through the surface subtended by Γ on both sides of this equation, one obtains

$$\oint_\Gamma \vec{B} \cdot \mathrm{d}\vec{l} = \mu_0 I + \mu_0 \varepsilon_0 \frac{\partial}{\partial t}(\Phi(t)) \tag{5}$$

where $\Phi(t)$ is the flux of the electrical field through the surface subtended by Γ. This electrical flux is easily calculated:

$$\Phi(t) = \frac{q(t)}{\varepsilon_0}(1 - \sin\alpha) \tag{6}$$

and equation (5) becomes

$$2\pi R|\vec{B}(C)| = \mu_0 I + \mu_0\varepsilon_0 \frac{\partial}{\partial t}\left[-\frac{q(t)}{\varepsilon_0}(1 - \sin\alpha)\right] = \mu_0 I \sin\alpha. \tag{7}$$

This is equivalent to equation (3) obtained with the Biot–Savart law.

5. Conclusions

The problem of the magnetic field is not a problem of magnetostatics but a time-varying fields situation. The Biot–Savart law gives the correct result because it is a general solution of Maxwell–Ampère equations, as can be easily seen by considering the rotation of the following expression:

$$\vec{B} = \frac{\mu_0}{4\pi}\vec{\nabla} \times \int_{\mathcal{V}} \frac{\vec{j}(\vec{r})}{r^2}\,\mathrm{d}^3\vec{r}. \tag{8}$$

By using the continuity equation $\partial\rho/\partial t + \vec{\nabla}\cdot\vec{j} = 0$, we obtain again the Maxwell–Ampère equation (4). This academic question is a nice illustration of the generality of the Biot–Savart law, and especially how it implicitly takes into account the charge conservation law.

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