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Scale Invariance and Scaling Exponents in Fully Developed Turbulence

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Abstract. — We apply the general formalism of equivalence of reference fields in scale invariant systems (Dubrulle and Graner, preceding paper [1]) to fully developed isotropic turbulence. Scale symmetry, the postulate of equivalence and regularity select the only physical solutions: the log-Poisson distribution, and its limits, the beta model and the Kolmogorov solution. The parameters left free by the symmetries are selected in relation with physical constraints applied on the turbulent flow. The link with previous models and experimental measurements of the scaling exponents is briefly discussed.

Résumé. — Nous appliquons le formalisme général de l'équivalence des repérages dans les systèmes invariants d'échelle (Dubrulle and Graner, article précédent [1]) à la turbulence développée isotrope. La symétrie d'échelle, le postulat d'équivalence et la régularité sélectionnent les seules solutions physiquement admissibles: la distribution log-Poisson, et ses limites, le modèle beta et la solution de Kolmogorov. Les paramètres laissés libres par les symétries sont sélectionnés en liaison avec les contraintes physiques imposées aux flots turbulents. Nous discutons brièvement le lien entre nos résultats, des modèles récents et des mesures expérimentales des exposants.

1. Introduction

1.1. MOTIVATIONS. — Over half a century, experimental progresses have motivated attempts to generalize the simplistic and celebrated Kolmogorov approach of turbulence. Basically, Kolmogorov tried to catch the essence of turbulence [2] with a very stringent scale invariance hypothesis: namely that the rate of energy dissipation is independent of the scale, in the inertial range where both the forcing and viscous dissipation are irrelevant. As pointed out by Landau [3] and noted by Kolmogorov himself [4], this hypothesis appears too strict to be

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realistic, and one can expect actual variations of the energy dissipation. This is believed to be the source of the deviations from the Kolmogorov prediction observed in increasingly accurate experimental measurements of the scaling exponents of the moments of velocity differences [4].

Among the various attempts to explain these deviations, the most successful often satisfy in a broader sense the scale invariance. Good examples are the multi-fractal model [5], or the She-Lévêque model [6], which was later shown to be equivalent to assuming a log-Poisson statistics for the energy dissipation [7, 8]. With this motivation in mind, we have developed in the preceding paper [1] a formalism for systems which are scale invariant in a statistical sense and obey a general principle of exponent relativity, generalizing the notion of equivalent systems of units. The aim of the present paper is now to apply this formalism to turbulence in order to constrain the scaling exponents.

Our argumentation proceeds as follows. In Section 1.2, we present well-known and newly derived scale symmetry properties of turbulence. We then use the classification of scaling exponents determined by general arguments in the preceding paper [1], using the same definitions and notations, and apply them to turbulence (Sect. 2). This selects the possible statistics for energy dissipation and velocity increments in the inertial range. We then discuss the link between our results, previous models and experimental measurements. (Sect. 3).

1.2. Scale Symmetry of Navier-Stokes Equations. — The Navier-Stokes equations are:

$$\begin{aligned} \partial_t \mathbf{u} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} &= -\boldsymbol{\nabla} p + \nu \nabla^2 \mathbf{u} + \mathbf{f}, \\ \boldsymbol{\nabla}\cdot\mathbf{u} &= 0, \end{aligned} \tag{1}$$

where t is the time, p the pressure, $\mathbf{u}$ the velocity field, ν the viscosity and $\mathbf{f}$ the external force. Turbulence [2] is defined as a statistically stationnary regime of the Navier-Stokes equations where an external forcing balances the viscous energy losses. However, it is often assumed that the scales ranging between the forcing and dissipation scales are correctly described by the force-free inviscid approximation ($f, \nu \to 0$): this "inertial range" increases with the Reynolds number Re [4]. In this limit, and taking into account the statistical stationnarity, the Navier-Stokes equations become invariant under the family of spatial dilations with arbitrary similarity exponent h and scale factor λ : $\mathbf{u} \to \lambda \mathbf{u}$, $\mathbf{x} \to \lambda^h \mathbf{x}$, where $\mathbf{x}$ is the space coordinate. We have not written the transformation rule for the pressure because it can be eliminated by the divergence-free condition (1b).

In applying the results derived in [1], we are led to consider two coarse grained fields at scale ℓ, the energy dissipation and the velocity gradients $\partial_x u_x$ (or, by isotropy, equivalently $\partial_y u_y$ or $\partial_z u_z$). Specifically, we consider a one-dimensional line with coordinate x, and define the energy dissipation at scale ℓ via [9]:

$$\epsilon_\ell(x) \;=\; \frac{1}{\ell}\int_{|x-x'|<\ell} \mathrm{d}x' \, \frac{1}{2}\, \partial_t u_i(x')\, u_i(x'), \tag{2}$$

and the velocity gradient $\partial_x u_x$ at scale ℓ:

$$\mathrm{d}u_\ell = \frac{1}{\ell}\int_{|x-x'|<\ell} x' \, \partial_x u_x(x'). \tag{3}$$

Note that the interest of considering velocity gradients averaged over size ℓ was already stressed by Eyink [10], using an analogy between turbulence and field theory. Only at the end of the calculations will we come back to the (longitudinal) velocity increments $\delta\mathbf{u}_\ell(\mathbf{x})$:

$$\delta\mathbf{u}_\ell(\mathbf{x}) = (\mathbf{u}(\mathbf{x}+\ell) - \mathbf{u}(\mathbf{x}))\cdot\frac{\boldsymbol{\ell}}{\ell}, \tag{4}$$

thanks to:

$$\delta u_\ell(\mathbf{x}) = \ell \mathrm{d}u_\ell. \tag{5}$$

The dilation symmetry then implies the scale symmetry for the energy dissipation or the velocity increments at scale ℓ, in the limit $f, \nu \to 0$:

$$\mathcal{S}_h(\lambda): \quad \ell \to \lambda\ell, \quad \mathrm{d}u \to \lambda^h \mathrm{d}u, \quad \epsilon \to \lambda^{3h+2}\epsilon, \tag{6}$$

with arbitrary scale factor λ and exponent h.

2. Scaling Exponents in the Inertial Range

2.1. Successive Moments. — We now consider the scale invariant range, i.e. the inertial range where the inviscid force-free approximation applies [4], in which $\mathrm{d}u_\ell$, δu_ℓ and ϵ_ℓ obey a power law:

$$\begin{aligned} \langle \epsilon_\ell^n \rangle &\propto \ell^{\tau(n)}, \\ \langle \mathrm{d}u_\ell^n \rangle &\propto \ell^{\chi(n)}, \\ \langle \delta u_\ell^n \rangle &\propto \ell^{n+\chi(n)} = \ell^{\xi(n)}. \end{aligned} \tag{7}$$

The scaling exponents $\xi(n)$ and $\tau(n)$ fully characterize the possible statistics of the velocity increments or energy dissipation, in the sense that prefactors are unimportant [11]. We now examine what constraints the scale symmetry imposes on the scaling exponents in turbulence, and therefore on the possible statistics.

2.1.1. Possible Statistics. — As discussed in [1], the scale symmetry amounts to a property of homogeneity in the "log-space" defined by the two translation-invariant variables $(\ln\langle \mathrm{d}u_\ell/\mathcal{R}\rangle, \ln\ell)$. Here, the "reference field" $\mathcal{R}$ is a scale invariant system of units, i.e. based on a power law of the scale ℓ. The same holds for the two translation-invariant variables $(\ln\langle \epsilon_\ell/\mathcal{R}\rangle, \ln\ell)$.

Reference [1] combines this "log-homogeneity" with a postulate of equivalence of scale invariant reference fields that generalizes the intuitive notion that systems of units are equivalent. The final result is that the only possible scaling exponents in a scale invariant system fall within three classes: a generic and a degenerate class characterized by divergence of high positive or negative moments; and a regular class corresponding to a log-Poisson statistics [7, 8]. All classes admit as limiting cases the Kolmogorov and the fractal (β-model) solutions.

2.1.2. Selection of log-Poisson. — These various classes correspond to a different geometry of the most intermittent structures in the system: there are two types of intermittent structures, with different codimension, in the generic case, while there is only one type of intermittent structure in the degenerate and log-Poisson case. This difference provides physical motivations for the selection of the possible scaling exponents in turbulence. Several numerical simulations [12] and experiments [13] indicate that the most intermittent structures in turbulence are only under the form of vortex filaments. This would already rule out the generic case. There is also a somewhat general consensus that moments in turbulence are regular although there are some claims [14] of evidence of moment divergence in geophysical flows, which however cannot be considered as isotropic homogeneous turbulence. If we accept both beliefs, we are left only with the log-Poisson case. The log-Poisson case therefore appears as *the only case compatible with the fundamental requirement of turbulence, namely the scale invariance, the geometrical constraints and the regularity of moments*. Its Kolmogorov or fractal limits are of course also acceptable.

2.1.3. The General Solution. — The energy dissipation is always positive. The velocity gradient can be either positive or negative. Keeping this difference in mind, we predict that χ, and τ take the log-Poisson type shape:

$$\begin{aligned}
\tau(n) &= n\Delta^{(\epsilon)} + C^{(\epsilon)}\left(1-(\beta^{(\epsilon)})^n\right),\\
\chi(n) &= \min_n\left[n\Delta_-^{(u)} + C^{(u)}\left(1-(\beta_-^{(\mathrm{du})})^n\right);\ n\Delta_+^{(u)} + C^{(u)}\left(1-(\beta_+^{(\mathrm{du})})^n\right)\right],
\end{aligned} \tag{8}$$

where C is the codimension of the most intermittent structures, Δ is the exponent characterizing their scaling properties, and β a yet free parameter, likely related to the conservation laws [15]. The subscript $+$ and $-$ label respectively the positive and the negative part of the velocity gradients. The superscripts (u) or (ϵ) have been explicited to recall that symmetry only predicts the shape of the scaling exponents, and not the values of the parameters, which may depend on the physical quantity. This expression calls for several remarks:

• the velocity increments, characterized by scaling exponents $\xi(n) = n + \chi(n)$ are also log-Poisson;

• if $\beta = 0$ one obtains the β model [16] $\tau(n) = n\Delta + C$;

• if $C = 0$, i.e. when the intermittent structures invade the whole space, one recovers the Kolmogorov limit $\xi(n) = n/3$, provided $\Delta^{(u)} = -2/3$.

The scaling exponents for the velocity increments and the energy dissipation each depend on seven parameters: $\Delta_\pm^{(u)}$, $\Delta^{(\epsilon)}$, $\beta_\pm^{(u)}$, $\beta^{(\epsilon)}$ and C. The case where both positive and negative gradients have the same scaling properties ($\Delta_+^{(u)} = \Delta_-^{(u)} \equiv \Delta^{(u)}$; idem for β) provides a significant simplification:

$$\begin{aligned}
\chi(n) &= n\Delta^{(u)} + C^{(u)}\left(1-(\beta^{(u)})^n\right),\\
\tau(n) &= n\Delta^{(\epsilon)} + C_-^{(\epsilon)}\left(1-(\beta^{(\epsilon)})^n\right).
\end{aligned} \tag{9}$$

We adopt this simplification from now on; if it turned out to be in contradiction with experimental observations, it could be relaxed at the price of more tedious computations.

2.2. Symmetry Constraints on the Scaling Exponents. — We may use one geometrical and three analytical constraints, to further restrict the range of possible parameters:

• whatever its definition, e.g. (2), ϵ is defined in term of velocity, so that it seems natural to assume that the geometry of their most intermittent structures is the same:

$$C^{(u)} = C^{(\epsilon)} \equiv C. \tag{10}$$

• the statistical stationarity implies conservation of energy:

$$\langle \epsilon_\ell \rangle = \epsilon_0, \tag{11}$$

where ϵ_0 is the input of energy per unit time and mass forced externally into the system. This means $\tau(1) = 0$ and, using (9):

$$\beta^{(\epsilon)} = 1 + \frac{\Delta^{(\epsilon)}}{C}. \tag{12}$$

• the scale symmetry (6) suggests that $\Delta^{(u)}$ and $\Delta^{(\epsilon)}$ are linked by:

$$\Delta^{(\epsilon)} = 3\Delta^{(u)} + 2. \tag{13}$$

• the Kolmogorov four-fifth law can be obtained exactly from the Navier-Stokes equations, in the inviscid limit [4]:

$$\langle(\delta u_\ell)^3\rangle = -\frac{4}{5}\epsilon_0\ell. \tag{14}$$

This implies $\xi(3)=1$, and thus, using (13) and (12):

$$(\beta^{(u)})^3 \;=\; 1+\frac{3\Delta^{(u)}+2}{C} \;=\; \beta^{(\epsilon)}. \tag{15}$$

To summarize these constraints, only two parameters suffice to quantify the turbulence: the codimension C of the most intermittent structures, and the "singularity" exponent for velocity increments $1+\Delta^{(u)}\equiv h_\mathrm{m}$. In term of these two parameters, the scaling exponents are then:

$$\begin{aligned}\chi(n) &= n(h_\mathrm{m}-1)+C\left(1-\beta^{n/3}\right),\\ \tau(n) &= n(3h_\mathrm{m}-1)+C\left(1-\beta^{n}\right),\end{aligned} \tag{16}$$

where:

$$\beta\equiv 1+\frac{3h_\mathrm{m}-1}{C}. \tag{17}$$

Note that (16) means that the exponents of the velocity increments follow

$$\xi(n)=nh_\mathrm{m}+C\left(1-\beta^{n/3}\right), \tag{18}$$

and are therefore linked to the exponents of the energy dissipation via:

$$\xi(n)=\tau(n/3)+n/3. \tag{19}$$

This relation, also known as the Kolmogorov refined similarity hypothesis [17], is usually obtained from a generalization of (14). Here it directly arises from the geometrical hypothesis (10) about the intermittent structures, independently of the values of the parameters. It can serve as a good test of the validity of our simplification (9). If for instance $\Delta_+\neq\Delta_-$ and/or $\beta_+\neq\beta_-$ for the velocity gradients, a reexamination of the conditions (10-14) taking into account (8) shows that one should observe a range of value of n for which this relation does not hold.

2.3. Values of the Parameters. — To be able to completely specify the scaling exponents in turbulence, one must assign some numerical values to the parameters h_m and C. We stress that these values can not be obtained from further symmetry considerations: as usual, symmetry only constraints the shape of the physical laws in a system, not the value of the corresponding parameters. The constants should then be computed from the Navier-Stokes equations, or measured in experiments or simulations.

2.3.1. Values of h_m. — We note that a restriction on the range of values for h_m occurs if one takes into account a result due to Frisch [4]: if the scaling exponents for velocity increments $\xi(n)$ are decreasing for large n, the incompressible approximation breaks down in the limit of infinite Reynolds number. For this reason, it is often considered that scaling exponents $\xi(n)$ should be a non-decreasing function of n. From equation (18), one may check that this is possible only if $h_\mathrm{m}\geq 0$, and if β, as defined in (17), is smaller than 1; i.e.:

$$0\leq h_\mathrm{m}\leq 1/3. \tag{20}$$

The upper bound corresponds to $\Delta^{(\epsilon)}=0$ and $\beta=1$, leading to the Kolmogorov solution $\xi(n)=n/3$ and $\tau(n)=0$. This solution corresponds therefore to case of minimal variance for the energy dissipation. In contrast, the lower bound $h_\mathrm{m}=0$ corresponds to a low variance of the velocity increments.

2.3.2. Values of C. — The conditions that C be the codimension of the intermittent structures imposes that C is positive, and smaller than the space dimension [1], i.e.:

$$0 \leq C \leq 3. \tag{21}$$

We are not aware of attempts to measure directly C. If we trust the visual appearance of the intermittent structures, which look like filaments, we get $C \approx 2$. However, Castaing [18] argues that the codimension of the intermittent structures is Reynolds dependent. This proves that β, h_{m} and C cannot be considered as universal and should be computed directly from the Navier-Stokes equations, or measured directly from experiments. This leads us to the problem of comparison of our model with existing data, and the derivation of possible stricter bounds on the parameters.

3. Discussion

Using symmetry considerations and geometrical and regularity properties of the Navier-Stokes equations, we were able to predict that both the energy dissipation and the velocity increments should follow a log-Poisson statistics, characterized by scaling exponents following (8). Using further analytical and geometrical constraints, we also showed that the value of the scaling exponents only depends on two parameters characterizing the most intermittent structures, their scaling exponent, h_{m} and and their codimension, C. The resulting two-parameters family of exponent includes for example the Kolmogorov 1941 model, for $h_{\mathrm{m}} = 1/3$ and $C = 0$, or the She and Lévêque model, designed for incompressible 3D isotropic turbulence, for $h_{\mathrm{m}} = 1/9$ and $C = 2$. It represents however only a sub-class of the multi-fractal models discussed in [5] or of the log-infinitely divisible models discussed in [8]. Our prediction is therefore more precise and could more easily be confronted with experimental data. Since we still keep two adjustable parameters, how can we experimentally confirm or infirm the log-Poisson statistics and/or determine the possible values of the parameters?

Using the technique of extended self-similarity, Benzi *et al.* [19] significantly improved the quality of the measurement of scaling exponents. These data are well fitted by the She and Leveque model. However, scaling exponents are a very bad way to discriminate between models. For instance, Nelkin [20] discusses a model based on another statistics proposed by Novikov [21], which is indistinguishable from She and Lévêque model up to $n = 100$. Discrimination is much better on first or second derivatives of the scaling exponents with respect to n [22]. Using this test, the log-Poisson law seems to be one of the best candidates to fit the GOY shell models of turbulence [22] and experiments at Reynolds number based on the integral scale $\mathrm{Re}_\lambda = 800$ [23].

Experimental measurements of the parameters of the log-Poisson statistics however raise the question of the universality of the parameters, a question already addressed in [7,8]. Numerical simulations on shell models apparented to turbulence indicate that the values of the parameters strongly depend on the conservation laws [15,24]. Even within the context of 3D incompressible turbulence, a Reynolds dependence of the parameters cannot be completely excluded. Fitted values of $\beta = 1 + (3h_{\mathrm{m}} - 1)/C$, namely 0.79 at $\mathrm{Re}_\lambda = 120$ [25] and 0.7 at $\mathrm{Re}_\lambda = 800$ [23, 26], are not exactly consistent with She-Lévêque values, but are compatible with our range. Other available measurements of the scaling exponents [27] can be fitted by a log-Poisson shape, with constants falling in our range (20)–(21) but different from the She and Lévêque values. Clearly, some additional work is needed to confirm and understand these variations and firmly assess the validity of the log-Poisson law.

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[11] Proof: consider a scale dependent process, e.g. δu_ℓ, such that in the inertial range $\langle \delta u_\ell^n \rangle = A_n \ell^{\xi(n)}$, where the A_n are some numerical prefactors. Let v be any *scale-independent* process, uncorrelated with δu_ℓ. The process $w_\ell = v \times \delta u_\ell$ has power-law moments $\langle w_\ell^n \rangle = B_n \ell^{\xi(n)}$ with $B_n = A_n \langle v^n \rangle$. Construct v so that $\langle w_\ell^n \rangle = \ell^{\xi(n)}$: therefore $\langle w_\ell^n \rangle = \langle w_\ell^2 \rangle^{\xi(n)/\xi(2)}$. In this case, the shape of the probability distribution is completely specified by the function $\langle w_\ell^2 \rangle$ and the value of the scaling exponents. For example, if w is log-Poisson, then $\xi(n) = \exp(n) - 1$.
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