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ASTRONOMY AND ASTROPHYSICS

Titius-Bode laws in the solar system

I. Scale invariance explains everything

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Abstract. According to the Titius-Bode law, the planetary distances to the sun follow a geometric progression. We review the major interpretations and explanations of the law. We show that most derivations of Titius-Bode law are implicitely based on the assumption of both rotational and scale invariance. In absence of any radial length scale, linear instabilities cause periodic perturbations in the variable $x = \ln(r/r_0)$. Since maxima equidistant in x obey a geometric progression in the variable r, Titius-Bode type of laws are natural outcome of the linear regime of systems in which both symmetries are present; we discuss possible non-linear corrections to the law. Thus, if Titius-Bode law is real, it is probably only a consequence of the scale invariance of the disk which gave rise to the planets.

Key words: planets and satellites: general – solar system: formation – hydrodynamics – instabilities

1. Introduction

1.1. Historical overview

Ever since Plato and until the recognition of the existence of chaos, the search for order and regularity in the universe has influenced many a physical theory or observation. The explanation of the distribution of planetary distances is a good illustration of this influence. In the greek antiquity, the first recorded attempts of setting a regularity in the sequence of ratios of successive planet orbit sizes seem to begin with Plutarque ($\sim$ 50 – 120 AD), who favored a pythagorean sequence of powers of 3 (Neugebauer 1975). Within a geocentric cosmology, from his $\sim$ 124 – 141 AD observations, Ptolemy (146) presents estimated ratios of successive planetary distances reinterpreted around 200 AD by Cassius Dio, linked with regular musical intervals. Hippolytus (230) claims that it is an heresy to imagine there could be no order in planet orbit intervals; he critically examines data attributed to Archimedes ($\sim$ 286 – 212 BC) to evidence alternating powers of 2 and 3, a platonician numerology Macrobius (400) also supports.

In modern times, the first theory within a heliocentric cosmology comes with Kepler. In his *Mysterium Cosmographicum* (1596), he assumed a construction where each planet orbit was a circle circumscribed to one of the five regular polyhedra: cube, tetra-, dodeca-, icosa- and octahedron (Crombie 1952). He then worked with Tycho Brahe to benefit from his cautious astronomical measurements, and was convinced that Mars' orbit is actually an ellipse; but even after he published his laws (Kepler 1609, 1619), he issued a late republication of the *Mysterium* sticking to the regular solids and spheres construction.

The search for an "order" in the solar system was revived in 1766 by Titius, who noticed that the known planet orbits (Mercury to Saturn) would follow a geometric progression provided a "missing" planet was inserted between Mars and Jupiter. Bode reformulated this observation in 1772 into the so called "Titius-Bode law", which Wurm expressed in 1787 under its more modern mathematical form:

$$r_n = 0.4 + 0.3 \times 2^n, \tag{1}$$

where the distance r_n of the n^{th} planet to the sun is expressed in astronomical units. The integer should take values $n = -\infty$ for Mercury, $n = 0$ for Venus, $n = 1$ for the Earth, and so forth. At that time, amidst a growing polemic, supporters of the "law" had such a strong belief that they searched for new planets at locations predicted by the progression. The discovery of Uranus in 1781 and of the first asteroid Ceres in 1801, both at locations close to prediction, reinforced their confidence into the "law". However, after the discovery of Pallas, a second asteroid, in 1802, Neptune in 1846 and Pluto in 1930 at locations not predicted by the law, its limitations became evident (Jaki 1972).

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1.2. Present status

Many modern researches attempted at modifying the Titius-Bode law in order to better fit the observed planetary distances using only positive integer values of n. The best fits finally seem to be based on geometric progressions, of the form:

$$r_n = r_0 K^n, \tag{2}$$

where r_0 is a normalizing distance and K a constant: for the solar system $K = 1.7$, $n = 1$ for Mercury, $n = 2$ for Venus, and so forth. Note that r_0 has no special significance, and was chosen only so that r_1 corresponds to the orbit of Mercury. A shift of r_0 by any factor K^m (m being any integer) just results in the shift in the numbering of planets. Similar laws were also searched in the orbits of satellites of the four major planets: K is respectively 1.6, 1.5 and 1.4 for the Jupiter, Saturn and Uranus system (see e.g. Neuhäuser & Feitzinger 1986). Such numbers should however not be taken too seriously, in view of the large degree of arbitrariness involved in derivations of "laws" of the type of Eq. (2). It is indeed customary to add one or two missing planets in the progression when needed, or to remove those presenting suspicious characteristics (eccentricity, inclination,...) which could be interpreted as evidences for capture or collision events. The geometric laws should therefore rather be viewed as a rough description of planetary and satellite distances, rather than "exact" laws (see Table 1 and Fig. 1).

Nottale (1992) presents an interesting Titius-Bode like law as consequence of long term behavior of chaotic systems. He obtains a quantification of possible orbits for planets, spaced as $r_n = r_0 n(n+1)$. However, the agreement of this law with observations is guaranteed by special assumptions such as the division of the solar system in two subsystems (inner and outer), with a different normalizing radius r_0.

1.3. Polemics

The general attitude towards the Titius-Bode laws (2) is twofold: skepticism or faith. Skeptic people argue that the "laws" are pure numerical coincidences and were produced by chance alone. For example, Lecar (1973) showed that approximate Titius-Bode laws can be generated by a sequence of random numbers subject to an excluded volume constraint, where adjacent planets cannot be "too close to each other". Faithful people use the fact that Titius-Bode laws are observed in the solar system and in satellite systems of giant planets to justify its possible physical significance. The law is then used as a constraint of theories of solar and satellite system formation and explanations are sought. A difficulty is to find a mechanism working in both solar and planetary systems, which are a priori rather different in nature (temperature, density, etc..). However, this difficulty does not seem to be a major limitation to the imagination of the theoreticians since there are over 15 explanations to the various forms of the Titius-Bode law.

In our opinion, this facility to produce a Titius-Bode law is even more puzzling than the reality of the law itself. It suggests there is a "hidden" order behind all the explanations and that most of them are in fact based on the same assumption. The purpose of this paper is to show that this basic assumption consists in both rotational and scale invariance. These two symmetries alone are sufficient to produce geometrical progressions in a suitably chosen variable.

The outline of our argumentation is as follow. In Sect. 2, we review some of the major theories of the Titius-Bode law and show that most of them are based on a consequence of the scale and rotational invariance, namely that the only characteristic length in the problem is the radial distance to the rotation axis. In Sect. 3, we use simple symmetry considerations to show qualitatively that in any scale invariant rotating system, the Titius-Bode law naturally arises in a linear regime; non-linear corrections to the law are discussed. Our conclusion follows in Sect. 4. In a companion paper (Dubrulle & Graner 1993, hereafter Paper II), we introduce an elegant method to solve scale-invariant problems and cook up an infinite number of Titius-Bode laws; we explicitly derive the simplest example, in a flat rotating gaseous disk.

2. Theories of Titius-Bode laws

2.1. Dynamical vs kinematical theories

Explanations to the Titius-Bode law can be divided in two categories. We refer as "dynamical" to the theories of the first type, which assume that the present law traces back to a period anterior or contemporary to planet formation; most of them describe instabilities occurring in the primordial protoplanetary disk, thus set constraints on its physical characteristics. Theories of the second category, called "kinematical", assume that the law physically originates from orbital interactions posterior to planet formation.

The pros and cons of each category have been discussed by Nieto (1972). He argues that observed deviations from the exact geometric law could be interpreted as the natural outcome of orbital evolution after planet formation. Those deviations have been quantified by Blagg (1913) and Richardson (1945) via the introduction of a periodic function in the original exponential law:

$$r_n = A(1.7)^n[B + f(\alpha + n\beta)]. \tag{3}$$

Here α and β are real constants, A and B are positive constants; f is a 2π periodic function, ranging between 0 and 1. All parameters depend on the satellite system under consideration (Solar System or giant planets). According to Nieto, f could be the result of the tendency to commensurabilities between the orbits. Such interpretation is somewhat favored by numerical simulations of Conway & Elsner (1988), which show that systems placed initially in Titius-Bode-like laws (increasing planetary distances) are very stable.

"Kinematical" theories sometimes contradict each other; for example, Molchanov (1968) explains the Titius-Bode law by resonances between the frequencies of the nine planets, while Souriau (1989) argues that the law is a consequence of the planets being as far of resonances as possible, to avoid catastrophic ejection events. Note however that kinematical theories based on resonances are probably all ruled out by the work of Hénon (1969), who showed that for any set of revolution periods, provided they are randomly selected using an "excluded volume" procedure, one can find resonances relations as good as those observed in the Solar System.

We are interested in explaining explanations of the Titius-Bode law, rather than the law itself. As it turns out, our discussion mainly applies to dynamical theories; we therefore focus on them, and do not comment further the relevance of one category or the other.

2.2. Brief overview of dynamical theories

A complete review of dynamical theories for the Titius-Bode law is given in Nieto (1972). They mainly assume one of four physical mechanisms to take place in the protoplanetary disk: planetesimal accretion; competition between gravity and electromagnetic forces; self-gravitational instability; and hydrodynamic or turbulent instabilities in the protoplanetary disk.

Accretion produces a Titius-Bode law via the "feeding zone" effect (Vityazev et al. 1977): during its rotation around the central object, a large planetesimal at a mean distance r sweeps an annular area of radial extent $2er$, where e is the eccentricity of the orbit. The non-overlapping of two adjacent "feeding zones", corresponding to the n^{th} and the $(n+1)^{th}$ planetesimal, then imposes:

$$r_n(1+e_n) = r_{n+1}(1-e_{n+1}), \tag{4}$$

where r_n and e_n are respectively the semi major axis and the eccentricity of the n^{th} planetesimal. The further assumption that all large planetesimals were formed with roughly the same eccentricity $< e >$ leads to a Titius-Bode law with a parameter:

$$K_{feed} = \frac{1+<e>}{1-<e>}. \tag{5}$$

The electromagnetic mechanism mainly relies on the dynamics of charged particles under mutual electrostatic influence in the solar gravitation field (Berlage 1930). Particles thus only experience $1/r^2$ force fields.

The gravitational instability mechanism, explicitly taking into account the cylindrical (rotational) symmetry of the disk, has been especially fashionable among Titius-Bode law builders. Prentice (1977) found a Titius-Bode law for the distribution of density maxima provided the collapse is homologous, i.e. self-similar; he was especially interested in the influence of turbulent convection.

In contrast, Polyachenko & Fridman (1972) showed that even in a cold disk, the instability leads to a Titius-Bode law provided that the surface density $\sigma(r) = \int \rho(r,z)dz$ varies with the radial distance as r^{-2}. When they realized that the initial protoplanetary dust sub-disk was probably not massive enough for their gravitational instability to take place, they turned to dissipative instabilities (Gor'kavyi et al. 1990). Again, a Titius-Bode law was found provided the height H of the disk is directly proportional to r and the turbulent viscosity scales like:

$$\nu_t \propto \frac{c_s^2(r)}{\Omega(r)} = c_s^2(r)\left(\frac{r^3}{GM_C}\right)^{1/2}, \tag{6}$$

where $\Omega(r)$ is the Keplerian frequency, M_C the mass of the central object (the sun or the giant planet), and c_s is the sound velocity. More recently, the dissipative instability scenario was reinvestigated by Willerding (1992), leading to rings following the Titius-Bode law; an underlying assumption is the constancy (r independence) of a Reynolds-type non-dimensional number:

$$\Re = \frac{sM_C}{2\pi\sigma\nu_t} = cte. \tag{7}$$

σ and ν_t are the surface density and turbulent viscosity, and s the growth rate of the instability.

Other theories were based on disk structure under fully developed turbulence, rather than linear instability. For instance, von Weisäcker (1948), then Kuiper (1951) showed that turbulence in the disk would lead to the creation of large vortices, organized in concentric circles. If the scale of the turbulence (the scale of the vortices) varies linearly with the radial distance, the frontier between the concentric vortex rings follows a Titius-Bode law.

2.3. Comments

Within such a tremendous diversity, these models share a common methodology. Each of them assumes a physical phenomenon to be the origin of planet formation. Then, coming to quantitative predictions, each model needs an hypothesis on the of course unknown physical properties of the primordial system. The most natural assumption, as long as we are totally ignorant, avoids introducing unnecessary parameters. Thus the first reflex is to propose a model with no supplementary length scale, other than the radial length scale r itself. Only Prentice seems to explicitly mention his hypothesis, and apparently no author points out its relevance. Such hypotheses include: homologous collapse, constant eccentricity, disk height H or vortex size $\propto r$, r^{-2} force fields, $\nu_t \propto c_s^2 r^{3/2}$, $\sigma \propto r^{-2}$, or $s/\sigma\nu_t = cte.$

This method seems reasonable; it is very popular, but simultaneously not innocent, for a very precise reason. Indeed, it never fails to produce a geometric progression in orbit sizes, even if not desired, and whatever the underlying physical model. For a "faithful" physicist, this prediction of a Titius-Bode law evidences the validity of the model and even offers quantitative constraints. We show in the next section why we believe that, in all the models we review, Titius-Bode laws arise from a same symmetry hidden in the equations, and not from physical phenomena they tend to modelize.

3. Rotating scale invariant systems and Titius-Bode laws

3.1. On symmetries

We want to point out that Titius-Bode laws arise in a whole class of physical problems; sufficient properties, by no mean necessary, are being (**P1**) invariant by rotation around an axis Oz, (**P2**) invariant by scale dilatation in the plane perpendicular to z. (**P1**) and (**P2**) set strong constraints along the vertical direction. There is no possible length scale along z other than r. The system is toroidal, with $|z|$ scaling like r; at extreme limits, it is either a flat disk, with all physical quantities vanishing outside the plane $z = 0$; or cylindrical, with a translational symmetry along z.

A particular case relevant to planetology, the flat disk, is considered in Paper II. We do not consider here problems which are also invariant under any in-plane translation, which fall under the conformal invariance analysis.

3.2. *Symmetry-breaking*

Two transformations can be defined by their action on the coordinates, and on vector or scalar quantities. The rotation of angle θ around the z axis leaves any scalar quantity invariant. It rotates the vector quantities by an angle θ along z and does not affect their modulus. The scale transformation stretches the radial coordinate r by a factor Λ, and any vector or scalar quantity g by a factor Λ^γ, where γ is an exponent depending on the set of equations and on the quantity. For example, in a rotating self-gravitating fluid, γ is respectively $3/2$, -3 or $-1/2$ for the time coordinate, the density or the velocity. Paper II presents an explicit definition and use of scale transformations. We consider a system described by a set of equations. If $g(r, \theta, z, t)$ is a physical field solution of the set of equations (e.g. density, vectorial velocity), then the equations are also satisfied by $TgT^{-1}(r, \theta, z, t)$, where T is either a rotation R_θ of arbitrary angle around the rotation axis, or a scale transformation S_Λ. The notation T^{-1} stands for the inverse of T: $R_\theta^{-1} = R_{-\theta}$ and $S_\Lambda^{-1} = S_{1/\Lambda}$.

For scalar quantities, the particular "scale invariant" solutions simply obey $g(r)/r^\gamma = cst$, while "rotational invariant" solutions are θ-independent. Generally speaking, g and its symmetric TgT^{-1} need not be the same, i.e. the individual solutions themselves need not be symmetric; on the other hand, the set of solutions itself must be globally invariant by any rotation or scale transformation.

Note that a single parameter can control the bifurcation, or spontaneous symmetry-breaking, from the particular case to the general one. Depending on the non-linearities (more precisely, on the sign of the lowest non-linear term) of the problem, such bifurcation is of "first-order" if the unstable solutions jump abruptly from the symmetrical one; and of "second-order" if the symmetry-breaking is a smoothly continuous function of the control parameter. For instance, in a flat self-gravitating disk, the ratio M_D/M_C of the disk mass to the mass of the central object is clearly such a control parameter: when $M_D/M_C \ll 1$, the disk is keplerian and stable; when $M_D/M_C \gg 1$, the disk is gravitationally unstable; the threshold in M_D/M_C is of the order of unity.

To simplify the demonstration we consider in this article only the subclass of the problems which are also (P3) time independent. A "solution" thus refers here to equilibrium states. In Paper II we relax this unessential constraint.

We define a linear problem (P4) as a problem in which all radial and azimuthal modes decouple. An important case of problem obeying (P4) is a second-order bifurcation just above the threshold value of the control parameter: (P4) simplifies the linear stability analysis around the equilibrium scale-invariant solution, otherwise tedious and completely computable only in limited cases (Yabushita 1966, 1969).

To decompose the azimuthal modes of a given physical quantity g it is natural to use the Fourier serie decomposition on the basis of the modes $\exp[i(m\theta + \theta_m)]$, where m is integer and θ_m real:

$$g(r, \theta) = Re(\Sigma_m a_m(r) \exp[i(m\theta + \theta_m)]), \tag{8}$$

where the symbol Re stands for real part. To perform the analogous decomposition for radial symmetry-breaking, we show hereafter why the most natural basis consists in the radial modes $\exp[ik\ln(r/r_0)]$ where r_0 and k are real constants; so that Titius-Bode laws arise in any problem obeying (**P1-2**) as well as the less essential (P3-4). This simply results from the rewriting of dynamical equations using the scale invariant variable $x = \ln(r/r_0)$, as we illustrate in Paper II. The following discussion emphasizes the generality of this treatment and its limitations.

3.3. *Generic equation*

Analytical functions of g which are invariant by rotation develop as series of the rotational invariant terms $(a_m)^p.(\bar{a}_n)^q$ where the bar denotes the complex conjugate, $(a_m)^p$ denotes the p^{th} power of the m^{th} azimuthal mode, and $mp = nq$. Analogously, analytical functions of g which are invariant by scale transformations develop over the various scale invariant terms such as g/r^γ or $r(\partial_r g)/g$, as well as terms built with functions of these, e.g. $r^{1-\gamma}\partial_r g$, or higher derivatives, e.g. $r^2(\partial_{r^2} g)/g$.

In a linear problem (P4), a single mode m is not coupled to any other mode and the quantities invariant by both rotation and scale dilatation are $r(\partial_r a_m)/a_m$, $a_m\bar{a}_m/r^{2\gamma}$, and so on. We call "generic" equations linking such invariants. One basic result of symmetric systems theory is that any physical equation describing an invariant problem can be written under a generic form (Gückenheimer & Holmes 1983). Then, as we now show, one equation linking g and $\partial_r g$ is a sufficient (and not necessary) condition for a_m to obey a Titius-Bode type of law.

Indeed, such an equation links a_m and $\partial_r a_m$. Its generic form is:

$$h_m\left(a_m\bar{a}_m/r^{2\gamma}, r(\partial_r a_m)/a_m\right) = 0, \tag{9}$$

where h_m is a function characteristic of the physical problem, implicitly linking these two quantities. If h_m is not singular, it can be inverted, and the second invariant written as a function of the first:

$$\partial_r(a_m) = \frac{a_m}{r} H_m\left(\frac{|a_m|^2}{r^{2\gamma}}\right), \tag{10}$$

where H_m is a regular function. With the new variables:

$$b_m = \frac{a_m}{r^\gamma}, \quad x = \ln r/r_0, \tag{11}$$

where r_0 is a normalizing radius, Eq. (10) takes a simpler form which evidences the scale-invariance:

$$\partial_x b_m = b_m G_m(|b_m|^2). \tag{12}$$

where the new function is simply $G_m = H_m - \gamma$. Equation (12), the simplest generic equation characterizing scale and rotational invariant problems, clearly indicates that the new Fourier coefficient b_m only depends on $x = \ln r/r_0$. Therefore, the initial physical quantity can be written under the form:

$$g(r, \theta) = r^\gamma \tilde{g}(x, \theta), \tag{13}$$

where the tilde denotes the scale-invariant equivalent function. Note that Eq. (12) is formally analogous to the generic equation governing bifurcation in rotating systems (see e.g. Knobloch 1993), with only our x dependence replacing his time dependence.

As long as the amplitude of the perturbation is small, the function G_m can be developed in Taylor serie:

$$G_m(|b_m|^2) = (\mu + ik) + (\eta + i\kappa)|b_m|^2 + ... \tag{14}$$

μ, k, η and κ are real coefficients. Using (14) and introducing the amplitude B_m and phase ψ_m of $b_m = B_m \exp[i\psi_m(x)]$, Eq. (12) writes:

$$\partial_x B_m = \mu B_m + \eta B_m^3 + O(B_m^5), \quad \partial_x \psi_m = k + \kappa B_m^2 + O(B_m^4). \tag{15}$$

The generic set of Eqs. (15) governs the phase and amplitude of any quantity g in a problem which satisfies (P1) to (P4): modes decoupling, time independence. The specificity of a given problem enters in the value of the constants μ, k, η and κ.

3.4. Linear laws

The linear regime of Eq. (15) is easy to study. Neglecting non-linear corrections, B_m and ψ_m can be found by straightforward integration. They are:

$$B_m = B_0 \exp[\mu x] = B_0 \left(\frac{r}{r_0}\right)^\mu, \quad \psi_m = kx = k \ln[\frac{r}{r_0}]. \tag{16}$$

Thus b_m writes:

$$b_m = B_0 \left(\frac{r}{r_0}\right)^\mu e^{ik\ln(r/r_0)} = B_0 \exp[(\mu + ik)x]. \tag{17}$$

From (17), it follows that the equal phase cylinders for a given mode m are spaced according to a Titius-Bode law. If r_n and r_{n+1} are the radial location of two consecutive cylinders, they obey the relation (2):

$$x_{n+1} = x_n + 2\pi/k = x_0 + 2n\pi/k, \quad r_{n+1} = r_n \exp[2\pi/k] = r_0 K^n. \tag{18}$$

where $K \equiv \exp[2\pi/k]$ is a constant, so that this geometric progression arises from the simple generic Eq. (9) we started with.

However, such law is lurking in any rotating scale invariant system. Therefore, if one studies a scale invariant system and assumes that planets are forming at the maxima (or minima) of one of the dynamical variable (the most popular being the density), he will automatically find a close Titius-Bode law within a linear development. The constant K of the law depends on the physics involved in the problem. For instance, in self-gravitating disk models, it mostly depends on the ratio between the mass of the central object, M_C, and the disk mass M_D (see Paper II for an illustration). An example of linear Titius-Bode law is displayed in Fig. 1. It corresponds to the Eq. (2) for the solar system, i.e. with r_0 = 0.21 A.U. and k = 11.8.

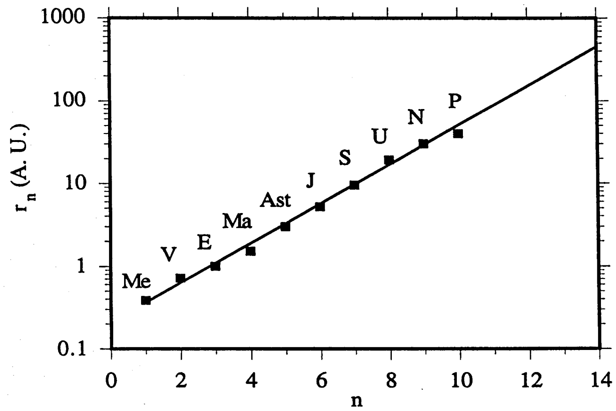

Fig. 1. Actual positions of planets in the solar system expressed in Astronomical Units (squares) and best fit by a linear Titius-Bode law (continuous line). See Table 1 for exact figures

3.5. Non-linear corrections

The non-linear corrections to (18) appear through the straightforward integration of the set of Eq. (15) up to the first non-linear terms, yielding:

$$B_m^2 = B_0^2 \frac{(r/r_0)^{2\mu}}{1 + \epsilon B_0^2 - B_0^2 \epsilon (r/r_0)^{2\mu}}, \quad \psi_m = k \ln[r/r_0] - \frac{\kappa}{2\eta} \ln\left[1 + B_0^2\epsilon - B_0^2 \epsilon (r/r_0)^{2\mu}\right], \tag{19}$$

where $\epsilon = \eta/\mu$. The large-r radial dependence of the amplitude and phase therefore depends on the sign of ϵ. If $\epsilon > 0$, both amplitude and phase diverge at $r = r_0[1 + 1/(\epsilon B_0^2)]^{1/(2\mu)}$; or, more physically, higher non-linear terms make this unphysical

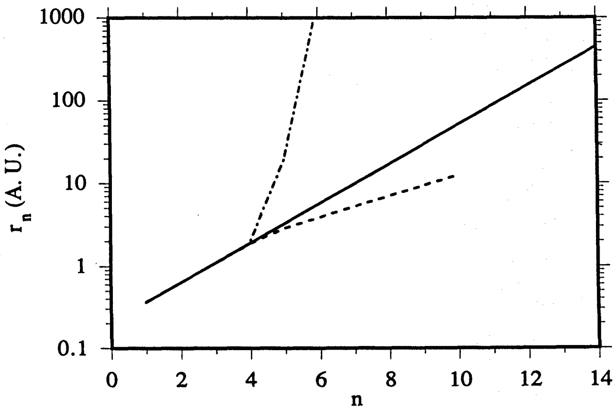

Fig. 2. Non-linear corrections to the linear Titius-Bode law (continuous line). According to the magnitude of κ, the non-linear law diverges (dashed dot line) or saturates at another amplitude (dashed line)

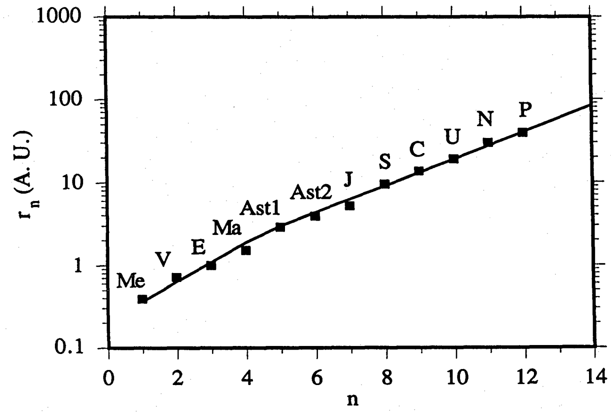

Fig. 3. Hand-made fit by a non-linear Titius-Bode law (continuous line) compared with observed positions in the total solar system (squares), which includes Chiron and two asteroid belts. See text for obtention of the fit and Table 1 for exact figures

Table 1. Position and mass of solar system objects: r_n is the mean distance, in A.U; M_n is the mass, in units of the Earth mass. The label obs, TB_{lin} and TB_{nl} correspond respectively to the observed distance, the best fit with a linear Titius-Bode law, and a hand-made non-linear fit. Ast1 and Ast2 refer to asteroid belts 1 and 2

Name	r_n/obs	M_n	r_n/TB_{lin}	r_n/TB_{nl}
Mercury	0.39	0.0553	0.36	0.36
Venus	0.72	0.8149	0.63	0.63
Earth	1	1	1.09	1.09
Mars	1.52	0.1074	1.88	1.88
Ast1	2.9	0.001	3.25	2.98
Ast2	3.9			4.33
Jupiter	5.2	317.938	5.63	6.28
Saturn	9.54	95.181	9.73	9.01
Chiron	13.70	?		13.18
Uranus	19.18	14.531	16.85	19.09
Neptune	30.10	17.135	29.15	27.65
Pluto	39.5	0.0022	50.43	40.05

divergence disappear. If $\epsilon < 0$, the amplitude saturates at the large-r limit:

$$B_m \to |\epsilon|^{-1/2}, \tag{20}$$

while the phase has the asymptotic form:

$$\psi_m \to (k - \kappa\epsilon)\ln[r/r_0]. \tag{21}$$

As a consequence, the non-linearities modify the Titius-Bode law in the spacing of surfaces of equal phase. In the case $\epsilon < 0$ where the non-linearities generate a saturation, the large-r Titius-Bode law is characterized by a constant different than at small r, namely $k - \kappa\epsilon$ (Fig. 2). Note that if $\kappa < -k|\epsilon|^{-1}$, ψ_m vanishes at an intermediate r value, void of oscillations, and changes sign at large r.

Planet formation is fundamentally a non-linear phenomenon, since it corresponds to a saturation (in the density amplitude). The present symmetry considerations offer no arguments in the debate whether planet positions were determined by linear or non-linear instabilities. For the Titius-Bode hunter, however, nonlinear laws introduce additional parameters; as an illustration, we show in Fig. 3 a flexible fit to the positions of the "total solar system" (see Table 1), including exotic objects such as Chiron. This fit even takes into account 2 asteroid belts between Mars and Jupiter, and is nothing more than numerology: since this list of the planets is of course arbitrary, we attach no physical significance to it.

The fit in Fig. 3 was obtained by recursively solving $\psi_0(r_n) = 2\pi n$ for various n. The phase function ψ_0 is:

$$\psi_0(r) = a\ln[r/r_0] + b\ln[1 + cr^{2\mu}]\,, \tag{22}$$

where the constant a, b, r_0, c and μ were obtained via an half-an-hour-juggling with numbers. They are $a = 2\pi/\ln[1.7] = 11.8$, $b = 0.55$, $r_0 = 0.21$, $c = 10^{-4}$ and $\mu = 4$. The phase (22) corresponds to the amplitude:

$$B_0(r) = B_0\frac{r^\mu}{(1 + cr^{2\mu})^{1/2}}\,. \tag{23}$$

If we assume planets form at positions related to local surface density maxima, B_0 is related to σ, the surface density, via:

$$\sigma(r) = r^s B_0(r)\exp[i\psi_0(r)] \tag{24}$$

with $s = -2$ in a scale-invariant system. The surface density (24) can be compared with the "actual" surface density of the solar system protoplanetary disk, of course unknown. Lacking reliable estimations, we arbitrarily plot in Fig. 4 a quantity $\sigma(r_n)$ varying like:

$$\sigma(r_n) \propto \frac{M_n}{2\pi r_n(r_n - r_{n-1})}, \tag{25}$$

in units of Earth mass/(A.U.)^2, where M_n is the mass of the n^{th} planet, r_n is its radial distance. The free normalizing constant B_0 of Eq. (23) has been chosen to get the best correspondence.

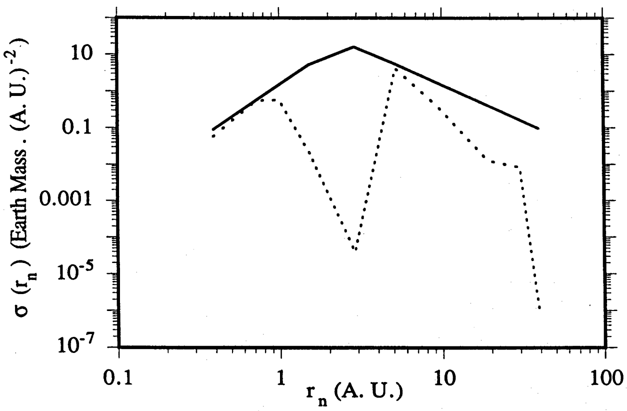

Fig. 4. Comparison of the surface density given by the non-linear Titius-Bode law (continuous line) and the actual surface density in the solar system (dashed line), computed from Table 1 and Eq. (25)

Obviously, the two surface densities differ sensibly, which is another indication of the model's lack of physical significance. Titius-Bode laws supporters could of course present various corrections; a classical argument is that the large dip in the solar system surface density is due to the weak mass of Mars and the asteroid belt, which are both under tidal influence of Jupiter, i.e. that the original surface density might therefore have been larger at their location.

4. Conclusion

We have shown the following results. (i) In any two-dimensional systems characterized by both rotational and scale invariance, the most natural Fourier basis vectors are $\exp[ik\ln(r/r_0)]$. (ii) Since gravitation respects both invariances, in a self-gravitating disk with no vertical length scale, extrema of density perturbations tend to follow a geometric progression of the form $r_n = r_0 K^n$. (iii) If the Titius-Bode laws of the solar system are more than pure numerological speculations, they may be simply interpreted as the signature of the scale and rotational invariance of the protoplanetary system. (iv) Neither symmetry considerations, nor observations of the solar system, can presently discriminate between linear and non-linear Titius-Bode laws.

We conclude that, if a model of planet or satellite formation leads to such geometric law for orbit diameters, this does not constitute a diagnostic of the model's validity, but only reflects its implicit scale and rotational invariance.

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